Introduction to pattern formation

Aerospace and mechanical engineering
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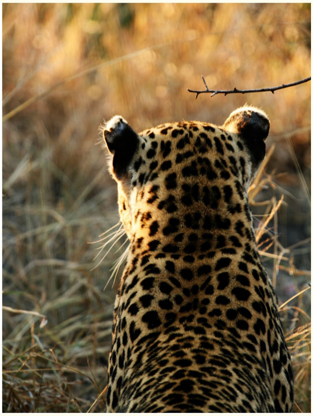
Natural patterns



Leopard spots

A herd of zebras

Slime-mold colony in early stages of aggregation

https://www.quantamagazine.org/biologists-home-in-on-turing-patterns-20130325/

Linear instability

- Turing instability
  - Reaction - diffusion equation
  - The Gierer - Meinhardt model.

- Rayleigh - Benard convection
- 1D Swift - Hohenberg equation
- Classification of linear instabilities

Nonlinear states

- Nonlinear stripe state of Swift - Hohenberg equation
- Stability balloons

[M. Cross and H. Greenside. 2009]

Turing instability

[Turing. 1952]

- At least two interacting chemicals are needed for pattern formation;
- Diffusion causes pattern forming instability
- Instability caused by diffusion causes growth at a particular wavelength.
- Pattern formation will not occur unless the diffusion coefficients of two reagents differ substantially

+ Reaction - diffusion equation

$$\begin{cases} \frac{\partial u_1}{\partial t} = f_1(u_1, u_2) + D_1 \frac{\partial^2 u_1}{\partial x^2} \\ \frac{\partial u_2}{\partial t} = f_2(u_1, u_2) + D_2 \frac{\partial^2 u_2}{\partial x_2^2} \end{cases}$$

u_i : concentration field of i

f_i : reaction rate of i (non-linear)

D_i : diffusion coefficient

Assume a Base state : $\begin{cases} f_1(u_{1b}\ u_{2b}) = 0 \\ f_2(u_{1b}\ u_{2b}) = 0 \end{cases}$

Linearize : perturbation u_{p_1}, u_{p_2} Assume mode : $e^{\sigma_q t} e^{iqx}$

$$\rightarrow \begin{cases} u_1(x,t) = u_{1b} + u_{p_1} \\ u_2(x,t) = u_{2b} + u_{p_2} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial u_{p_1}}{\partial t} = \left.\frac{\partial f_1}{\partial u_1}\right|_b u_{p_1} + \left.\frac{\partial f_1}{\partial u_2}\right|_b u_{p_2} + D_1 \frac{\partial^2 u_{p_1}}{\partial x^2} \\ \frac{\partial u_{p_2}}{\partial t} = \left.\frac{\partial f_2}{\partial u_1}\right|_b u_{p_1} + \left.\frac{\partial f_2}{\partial u_2}\right|_b u_{p_2} + D_2 \frac{\partial^2 u_{p_2}}{\partial x^2} \end{cases}$$

write in vector form :

$$\Rightarrow \frac{\partial \vec{u}_p}{\partial t} = A\vec{u}_p + D\frac{\partial^2 \vec{u}_p}{\partial x^2}, \quad A = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix}_b, \quad D = \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix}$$

$\Rightarrow$ Eigenvalue problem: $\sigma_q \vec{u}_q = A_q \vec{u}_q$, $\quad A_q = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} - D_1 q^2 & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} - D_2 q^2 \end{bmatrix}$

A particular solution : $(c_1 \vec{u}_{q_1} e^{\sigma_{q_1} t} + c_2 \vec{u}_{q_2} e^{\sigma_{q_2} t}) e^{iqx}$

→ characteristic polynomial : $\det(A_q - \sigma_q I) = 0$

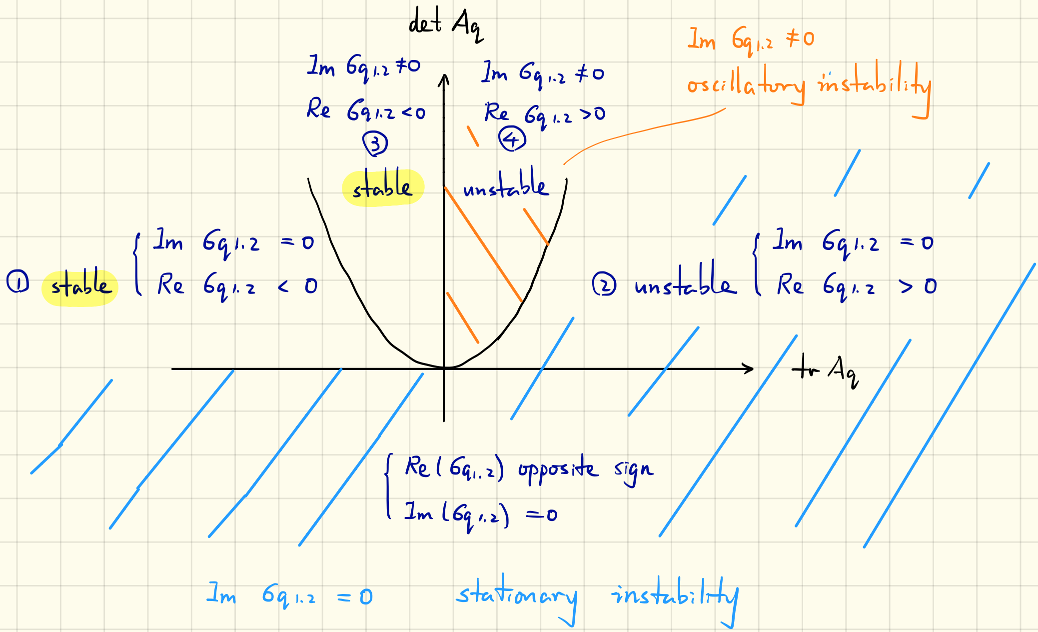
$$\sigma_q^2 - (\mathrm{tr} A_q)\,\sigma_q + \det A_q = 0$$

→ 2 eigenvalues : $\sigma_{q1,2} = \dfrac{\mathrm{tr} A_q \pm \sqrt{(\mathrm{tr} A_q)^2 - 4 \det A_q}}{2}$

$$\sigma_{q1,2} = \frac{\mathrm{tr} A_q \pm \sqrt{(\mathrm{tr} A_q)^2 - 4 \det A_q}}{2}, \quad \begin{cases} \det A_q = \sigma_{q1} \sigma_{q2} \\ \mathrm{tr}\, A_q = \sigma_{q1} + \sigma_{q2} \end{cases}$$

+ $\det A_q < 0$: $\sigma_{q1,2}$ are real with opposite signs
+ $\det A_q > 0$:

$$\begin{cases} (\mathrm{tr} A_q)^2 - 4 \det A_q > 0 & \begin{cases} \mathrm{tr} A_q < 0 \ (\sigma_{q1,2} < 0) & \text{stable } ① \\ \mathrm{tr} A_q > 0 \ (\sigma_{q1,2} > 0) & \text{unstable } ② \end{cases} \\ (\mathrm{tr} A_q)^2 - 4 \det A_q < 0 & \begin{cases} \mathrm{tr} A_q < 0 & \text{stable } ③ \\ \mathrm{tr} A_q > 0 & \text{unstable } ④ \end{cases} \end{cases}$$



+ stable : $Re(\sigma_{q_{1,2}}) < 0$.

$$\begin{cases} tr\, A_q < 0 \\ \det(A_q) > 0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_2} - q^2(D_1 + D_2) < 0 \\ \left(\frac{\partial f_1}{\partial u_1} - q^2 D_1\right)\left(\frac{\partial f_2}{\partial u_2} - q^2 D_2\right) - \frac{\partial f_1}{\partial u_2}\frac{\partial f_2}{\partial u_1} > 0 \end{cases}$$

$D_1\ D_2\ q^2$ are positive $\Rightarrow \frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_2} < 0$

If no diffusion, $\frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_2} < 0$ always true.

$\Rightarrow$ diffusion destabilize : $\det(A_q)$ change to negative

$$\rightarrow \text{find} \begin{cases} q_m^2 = \dfrac{f_{2u_2} D_2 + f_{1u_1} D_1}{2 D_1 D_2} \\ \min(\det A_{q_m}) = f_{1u_1} f_{2u_2} - f_{1u_2} f_{2u_1} - \dfrac{(f_{1u_1} D_2 + f_{2u_2} D_1)^2}{4 D_1 D_2} \end{cases}$$

Set $\det(A_{q_m}) < 0 \rightarrow \underbrace{D_1 f_{2u_2} + D_2 f_{1u_1} > 2\sqrt{D_1 D_2 (\underbrace{f_{1u_1} f_{2u_2} - f_{1u_2} f_{2u_1}}_{>0})}}_{\text{condition for linear instability}}$

$\left(f_{iu_j} = \frac{\partial f_i}{\partial u_j} \right)$

$\Delta \Rightarrow D_1 f_{2u_2} + D_2 f_{1u_1} > 0$, with $f_{1u_1} + f_{2u_2} < 0$

$\Rightarrow f_{1u_1}$ and f_{2u_2} have opposite signs

Set $f_{1u_1} > 0$, $f_{2u_2} < 0$

define diffusion length : $l_1 = \sqrt{\frac{D_1}{f_{1u_1}}}$, $l_2 = \sqrt{\frac{D_2}{-f_{2u_2}}}$

$\rightarrow q_m^2 = \frac{1}{2}\left(\frac{1}{l_1^2} - \frac{1}{l_2^2}\right)$

$\rightarrow l_1 < l_2$

$$\Rightarrow \begin{cases} f_{1u_1} > 0 \\ f_{2u_2} < 0 \end{cases} \Rightarrow \begin{cases} \text{chemical 1 enhances it own instability : activator} \\ \text{chemical 2 inhibits ——— inhibitor} \end{cases}$$

$l_2 > l_1 \Rightarrow$ local activation with long-range inhibition

$\frac{D_2}{D_1} > -\frac{f_{2u_2}}{f_{1u}}$

The Gierer - Meinhardt model.

[Gierer and Meinhardt 1972]

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \rho \dfrac{u_1^2}{u_2} - \mu_1 u_1 + D_1 \dfrac{\partial^2 u_1}{\partial x^2} \\ \dfrac{\partial u_2}{\partial t} = \rho u_1^2 - \mu_2 u_2 + D_2 \dfrac{\partial^2 u_2}{\partial x^2} \end{cases}$$

μ_1, μ_2: decay rate

Simplify by : $\rho = 1$, $\mu_2 = 1$, $\rightarrow \begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{u_1^2}{u_2} - \mu_1 u_1 + D_1 \dfrac{\partial^2 u_1}{\partial x^2} \\ \dfrac{\partial u_2}{\partial t} = u_1^2 - u_2 + D_2 \dfrac{\partial^2 u_2}{\partial x^2} \end{cases}$

Find a base state $u_{1b} = \frac{1}{\mu_1}$, $u_{2b} = \frac{1}{\mu_1^2}$

Jacobian : $A = \begin{pmatrix} \frac{2u_1}{u_2} - \mu_1 & -\frac{u_1^2}{u_2^2} \\ 2u_1 & -1 \end{pmatrix}_{(u_{1b}, u_{2b})} = \begin{pmatrix} \mu_1 & -\mu_1^2 \\ \frac{2}{\mu_1} & -1 \end{pmatrix}$

stable: $\begin{cases} \mathrm{tr} A < 0 \\ \det A > 0 \end{cases} \Rightarrow \begin{cases} \mu_1 < 1 \\ \mu_1 > 0 \end{cases} \Rightarrow 0 < \mu_1 < 1$

Add diffusion and small perturbation u_{1p} u_{2p}

$$\rightarrow \begin{cases} \dfrac{\partial u_{1p}}{\partial t} = \mu_1 u_{1p} - \mu_1^2 u_{2p} + D_1 \dfrac{\partial^2 u_{1p}}{\partial x^2} \\ \dfrac{\partial u_{2p}}{\partial t} = \dfrac{2}{\mu_1} u_{1p} - u_{2p} + D_2 \dfrac{\partial^2 u_{2p}}{\partial x^2} \end{cases}$$

eigenvalue problem $\frac{\partial}{\partial t}\begin{pmatrix} u_{1p} \\ u_{2p} \end{pmatrix} = \begin{pmatrix} \mu_1 - q^2 D_1 & -\mu_1^2 \\ \frac{2}{\mu_1} & -1 - q^2 D_2 \end{pmatrix}\begin{pmatrix} u_{1p} \\ u_{2p} \end{pmatrix}$

+ Looking for at least one $\mathrm{Re}(\sigma_q) > 0$

$\rightarrow \det A_q < 0 \Rightarrow (\mu_1 - q^2 D_1)(-1 - q^2 D_2) + 2\mu_1 < 0$

$\Rightarrow -D_1 + \mu_1 D_2 > 2\sqrt{D_1 D_2 \mu_1}$

Set periodic boundary condition:

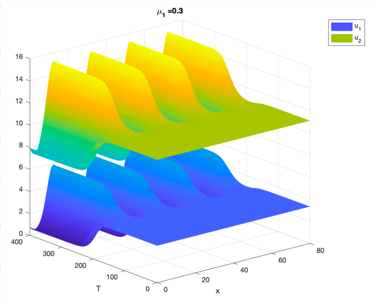
$\rightarrow$ solution: $\sum_n A_n e^{\sigma_q t} \cos(qx)$, $q = \frac{n\pi}{L}$, $n = 1, 2, 3 \cdots$

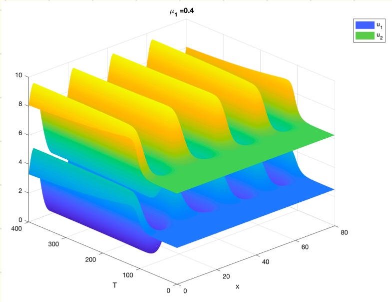
$$q_{min}^2 = \frac{\mu_1 D_2 - D_1 + \sqrt{(\mu_1 D_2 - D_1)^2 - 4\mu_1 D_2 D_1}}{2D_1 D_2} = \frac{\pi^2}{L^2}$$

$\rightarrow$ critical length $L_c = \frac{\pi}{q_{min}}$

✔

$\mu_1 = 0.3$. $D_1 = 1$. $D_2 = 30$ $\qquad$ $\mu_1 = 0.4$, $D_1 = 1$, $D_2 = 30$

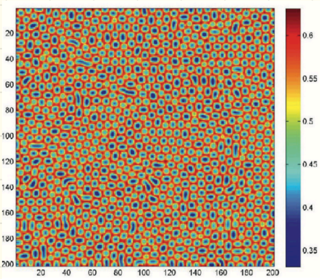


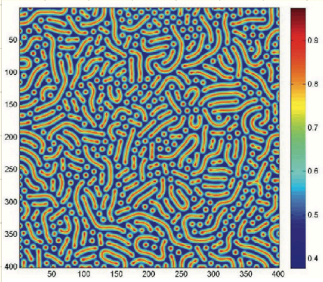


[Y. Song. et al 2017]

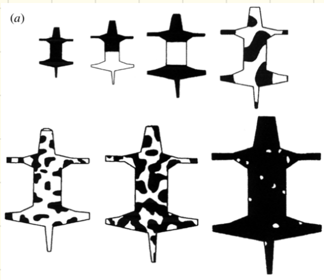
$$
\begin{cases}
\dfrac{\partial u_1}{\partial t} = r\left[\dfrac{u_1^2}{(1+pu_1^2)u_2} - cu_1\right] + D_1\nabla^2 u_1 \\[2ex]
\dfrac{\partial u_2}{\partial t} = r(u_1^2 - au_2) + D_2\nabla^2 u_2
\end{cases}
$$

After 50000 iterations, inhibitor: in different $\frac{a}{c}$





[J.D. Murray 2012]



numerical simulation.

only scale parameters are different

Reaction-diffusion models

[A.R. Sanderson. et.al 2006]

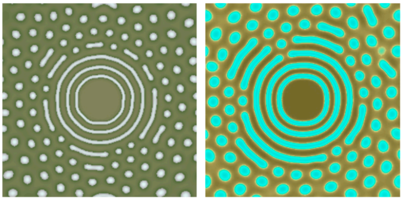


Figure 12: *Using radially gradient growth factor,* $\beta = f(r) \pm 0.1\%$ *and reaction rate,* $s = g(r)$ *to form a circular stripe-spot pattern.*

Figure 14: *Blue Spotted Puffer Fish (Arothron caeruleopunctatus) found in the Indo-Pacific region. Image courtesy of Jeffrey Jeffords/divegallery.com.*

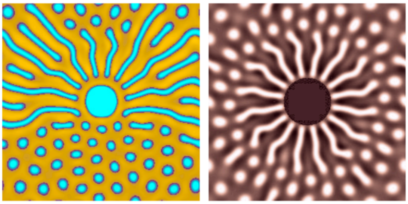
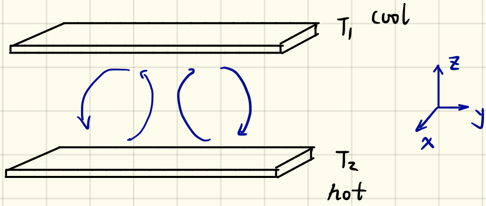


Figure 15: *Using radially gradient growth factor,* $\beta = h(\theta) \pm 0.1\%$ *and reaction rate,* $s = g(r)$ *to form a radial stripe-spot pattern.*

Figure 17: *Map Toby Puffer Fish (Arothron mappa) found in the Indo-West Pacific region. Image courtesy of Massimo Boyer/edge-of-reef.com.*

Rayleigh - Benard convection



conduction only → convection
→ unstable

$|T_0 - T_H|$ critical. value ?

+ Governing equation,

Cont. $\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = \frac{\partial \rho}{\partial t}$

Mom. $\begin{cases} \rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial y}} + \cancel{v\frac{\partial u}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \mu\left(\cancel{\frac{\partial^2 u}{\partial y^2}} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right) + \cancel{\rho g_y} \\ \rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial y}} + \cancel{v\frac{\partial v}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \mu\left(\cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right) + \underline{\rho g_z} \end{cases}$

gravitational force vertical

Tem. $\rho C_p\left(\cancel{\frac{\partial T}{\partial t}} + \cancel{u}\frac{\partial T}{\partial y} + \cancel{v}\frac{\partial T}{\partial z}\right) = k\left(\cancel{\frac{\partial^2 T}{\partial y^2}} + \frac{\partial^2 T}{\partial z^2}\right)$

+ To find steady state (stationary fluid)

+ Boussinesq approximation

$\rho(T)$, $\begin{cases} \text{only in "}\rho g\text{" term.} \\ \text{const for other terms} \end{cases}$ Assume: $\rho = \rho_0[1 - \alpha(T - T_1)]$, $\alpha > 0$

+ stationary fluid. $u=0$, $v=0$

$\Rightarrow \begin{cases} \frac{\partial P}{\partial y} = 0 \\ \frac{\partial P}{\partial z} = -g\rho_0[1 - \alpha(T - T_1)] \\ k\frac{\partial^2 T}{\partial z^2} = 0 \end{cases} \Rightarrow \begin{cases} U_0 = 0 \\ T_0 = T_1 - \frac{T_2 - T_1}{H} \\ P_0 = P_2 - gP_2\left(z + \alpha\beta\frac{z^2}{2}\right) \end{cases}$

+ Add small perturbation, linearize equation

$\begin{cases} \nabla\cdot\vec{u}_p = 0 \\ \frac{\partial \vec{u}_p}{\partial t} = -\frac{1}{P_2}\nabla P_p + \alpha g T_p \vec{k} + \nu\nabla\vec{u}_p \\ \frac{\partial T_p}{\partial t} - \beta v_p = k\nabla^2 T_p \end{cases}$

+ Dimesionless equation,

$\begin{cases} \nabla\cdot\vec{u} = 0 \\ \left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} = -\nabla P + R_a P_r\vec{k} + P_r\nabla^2\vec{u} \\ \left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)T = \vec{k}\cdot\vec{u} + \nabla^2 T \end{cases}$

scaling:
length H
time $\frac{k}{H^2}$
temperature: $T_2 - T_1$

Rayleigh number $Ra = \frac{\alpha g (T_2 - T_1) H^3}{k\nu}$

Prandtl number $Pr = \frac{\nu}{k}$

Boundary conditions. $v_p(z=0,\ z=H) = 0$, $T_p(z=0,\ z=H) = 0$

stress free. $v_p = 0$, $\frac{d^2 v_p}{dz^2} = 0$, $\frac{d^4 v_p}{dz4} = 0$

→ Eigenvalue problem → eigenvalues → unstable state.

(find critical situation)

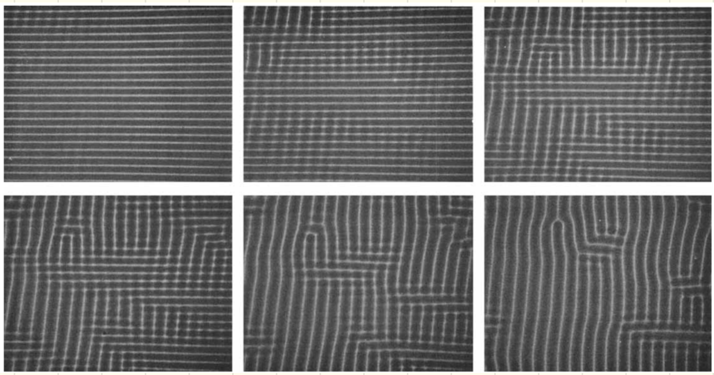
[F. H. Busse and J. A. Whitehead 1971]

$z = \frac{1}{2} H$

Ra as control parameter

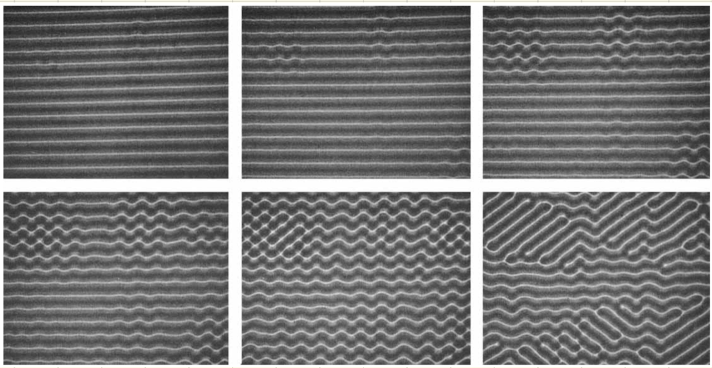
Find critical $Ra_c = 1708$

convection rolls



cross-roll instability

$Ra = 3000$

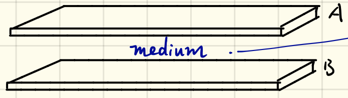


zigzag instability

$Ra = 3600$

Pattern - forming : (by instability)

confined direction $\vec{x}_{\parallel}$



is driven out of equil. by ∇T, v, ...

$\vec{x}_{\perp}$ extend direction (pattern formation)

+ For linear stability analysis :
  1. eliminate physical lateral boundaries (ex: set infinite boundary or periodic boundary)
  2. the system translationally invariant in extended directions.
  3. identify stationary uniform nonequil. states as starting point for pattern formation.

+ Linearized problem (evolution of tiny perturbation of uniform state)

$\rightarrow \exp(\sigma_{\vec{q}}\, t) \exp(i\vec{q}\cdot\vec{x}_{\perp})$

$\vec{q}$: wave vector

$\sigma_{\vec{q}}$: growth rate of perturbation (complex value) ($\vec{q}$ - dependent)

$\rightarrow$ linearly stable : $Re(\sigma_{\vec{q}}) < 0$, all small perturbation decays to zero

linearly unstable : $Re(\sigma_{\vec{q}})$ first becomes positive at critical parameter value.

+ linear instability :

Critical wave vector : $Max[Re(\sigma_{\vec{q}})]$ first becomes positive.

Critical wave number magnitude q_c (length scale of growing perturb. $\frac{2\pi}{q_c}$)

Critical frequency $\omega_c = -Im(\sigma_{q_c})$ $\quad [e^{i(q_c x - \omega_c t)}]$

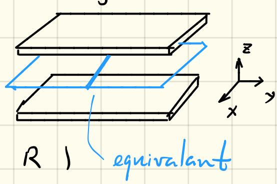
if $\omega_c = 0$: stationary type of instability

if $\omega_c \neq 0$: oscillatory " " "

One dimentional Swift-Hohenberg equation [Swift and Hohenberg 1977]

$$\frac{\partial u}{\partial t}(x,t) = (r-1)u - 2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - u^3$$

$$\frac{\partial u}{\partial t}(x,t) = ru - \left(\frac{\partial^2}{\partial x^2}+1\right)^2 u - u^3$$



r : control parameter (ex : Rayleigh number R)

x : extended coordinate

+ One solution : $u=0$: zero velocity conduction state

Critical value r_c when $u=0$ unstable (perturb → grow exp)

+ Linear stability analysis

$u_p(x,t) = u(x,t) - u_b$, arbitrary nearby solution $u(x,t)$

$$\rightarrow \frac{\partial u_p}{\partial t} = \left\{(r-1)[u_p+u_b] - 2\frac{\partial^2}{\partial x^2}[u_p+u_b] - \frac{\partial^4}{\partial x^4}[u_p+u_b] - [u_p+u_b]^3\right\}$$

$$-\left\{(r-1)u_b - 2\frac{\partial^2}{\partial x^2}u_b - \frac{\partial^4}{\partial x^4}u_b - u_b^3\right\}$$

$$= (r-1)u_p - 2\frac{\partial^2}{\partial x^2}u_p - \frac{\partial^4}{\partial x^4}u_p - \cancel{u_p^3} + \cancel{3u_p^2 u_b} - \cancel{3u_p u_b^2}$$

infinitesimal perturb. $u_b = 0$

$$\Leftrightarrow \frac{\partial u_p}{\partial t} = \left[r-1-2\frac{\partial^2}{\partial x^2} - \frac{\partial^4}{\partial x^4}\right]u_p \quad \text{—— linear}$$

- Assume particular solution, $u_p(x,t) = Ae^{\sigma t}e^{\alpha x}$

$$\Rightarrow \sigma = r-1-2\alpha^2-\alpha^4$$

α : ① if infinitely large boundary : $\begin{cases}\text{consistent with uniform } u_b \text{ in space} \\ \text{not consistent with } u_p\text{, unless } \alpha = \text{imag.}\end{cases}$

② if periodic boundary : $\begin{cases}\text{consistent with } u_b\text{.} \\ u_p(x,t) = u_p(x+L,t) \\ e^{\alpha x} = e^{\alpha(x+L)}\end{cases}$

$\rightarrow e^{\alpha L} = 1$, $\alpha L = 2\pi i m$ (so that α pure imag.)

$\alpha = qi$, restrict q to infinitely quantized values :

$$q = m\left(\frac{2\pi}{L}\right), \quad m = 0, \pm1, \pm2 \cdots$$

- Thus, ①&② consistent with $\begin{cases}\text{uniform base } u_b \\ \text{single exponential mode. } u_p = Ae^{\sigma t}e^{iqx}\end{cases}$

Linearized evolution equation can be solved by a single exp. mode.

△ growth rate: $\sigma_q = r - (q^2-1)^2$ small-amp spacially periodic perturb $\alpha = qi$ will grow/decay exp. in time with σ_q.

△ General solution
$$\begin{cases} u_p(x,t) = \sum_q C_q e^{\sigma_q t} e^{iqx} & \text{periodic Boundary} \\ u_p(x,t) = \int_{-\infty}^{\infty} C_q e^{\sigma_q t} e^{iqx}\, dq & \text{infinite Boundary} \end{cases}$$

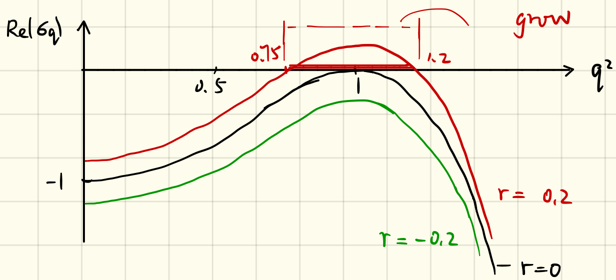
→ $\max \mathrm{Re}(\sigma_q) < 0$ $(r<0)$ ⇒ $u_b = 0$ linearly stable

+ Growth rates and instability diagram.

want to determine when $\max_q \mathrm{Re}(\sigma_q)$ change from (−) to (+)

$\sigma_q = r - (q^2-1)^2$ ⇒ $\max_q \mathrm{Re}(\sigma_q)$ when $q = 1$

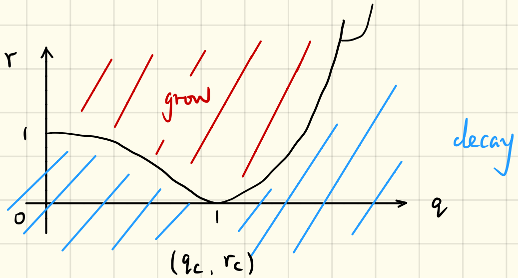
△ $r < 0$: linearly stable
$r > 0$: linearly unstable
$r_c = 0$: $\mathrm{Re}(\sigma_q)$ first attain positive $q_c = 1$



△ if initial u_p is small-amp. noise all Fourier coefficients non zero.
→ cellular pattern will start to grow

characteristic scale of perturbation: $\frac{2\pi}{q_c}$

△ neutral stability curve [$Re(\sigma_q) = 0$, $r = (q^2-1)^2$]



△ Thus, r just larger than r_c, expect cellular parttern will grow with wave num. q_c

+ Steps of linear stability analysis

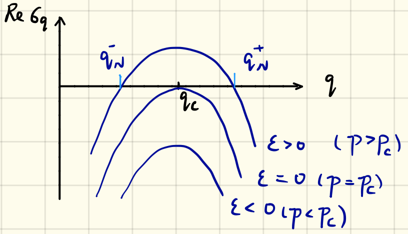
1. Obtain explicite evolution eq.
2. dimensionless equation
3. Replace boundary with infinite / periodic
4. find one time-independent uniform state u_b (with respect to x extend)
5. Linearize eq. about u_b, infinitesmal u_p.
   Coefficients not depend on $x_\perp$ nor time
6. use a particular solution $u_p = u_{\vec{q}}(x_\parallel) e^{\sigma_q t} e^{i\vec{q}\cdot\vec{x}_\perp}$ solve linearized eq.
   → wave vector dependent growth rate $\sigma_{\vec{q}}$
7. Analyze $Re(\sigma_{\vec{q}}) - \vec{q}$
8. Map out linear stability of uniform states as function of parameters.
   by repeating (4.-7) for different parameter vector $\vec{p}$;
   identify p_c $\max Re(\sigma_{\vec{q}}) = 0$

Classification of linear instabilities

- control parameter p [r in swift-Hohenberg eq. / Ra in Rayleigh-benard convection]

  reduced control parameter $\varepsilon = \frac{p - p_c}{p_c}$

- Type I instability [instability occurs $p > p_c$]



unstable to perturbation over a band of wave numbers
$q_N^- < q < q_N^+$

expand at q_c, $\sigma_{q_c} \approx \frac{1}{\tau_0}\varepsilon$
expand max $\mathrm{Re}(\sigma_q) = -\frac{\xi_0^2}{\tau_0}(q-q_c)^2$

$\rightarrow \quad \sigma_q \approx \frac{1}{\tau_0}[\varepsilon - \xi_0^2 (q-q_c)^2]$

Type I-s : stationary instability (standing wave)
Type I-o : oscillatory instability (traveling wave)

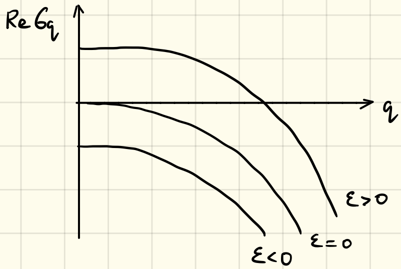
- Type II instability [growth rate is always zero at $q = 0$]

$\mathrm{Re}\,\sigma_q \approx D(\varepsilon q^2 - \frac{1}{2}\xi_0^2 q^4)$ D: dimension of diffusion const.

- characteristic length $\frac{2\pi}{q_c} \rightarrow \infty$ when $\varepsilon \rightarrow 0^+$

Type II-s
Type II-o

△ Type III instability (max $\mathrm{Re}\,\sigma_q$ at $q=0$)



$$\mathrm{Re}\,\sigma_q \approx \frac{1}{\tau_0}\left[\varepsilon - \xi_0^2 q^2\right]$$

→ spacial structure on a large length scale

$\left\{ \begin{array}{l} \text{Type III - s} \\ \text{Type III - o} \end{array} \right.$

Non-linear states

- Nonlinearities in the evolution equations generates spacial harmonics.
  - - leads to $\begin{cases} \text{steady spacially periodic sol. for stationary inst.} \\ \text{nonlinear oscillation or waves for oscillatory inst.} \end{cases}$

+ Nonlinear saturation

Saturated nonlinear steady state:

Nonlinearity can cause a time-independent state such that nonlinear terms have same magnitude as linear terms and balance them.

- If p slightly above p_c, small nonlinear term sufficient to balance small linear growth rate.
  Thus, stationary solution grows as p increase
  Supercritical bifurcation (forward bifurcation) second-order transition.
- If nonlinear term enhance growth rate initially.
  even though p slighty larger than p_c, disturbance will grow to a large value.
  Subcritical bifurcation (backward bifurcation) first-order transition.

+ Complex amplitude

- A growing solution: $\vec{u}(\vec{x},t) = A(t)\, e^{i\vec{q}\cdot\vec{x}_\perp}\, \vec{u}_q + A^* e^{-i\vec{q}\cdot\vec{x}_\perp}\, \vec{u}_q^*$

$\begin{cases} A(t): \text{Time dependent amplitude of perturbation} \\ \vec{u}_q^*,\ A(t)^*: \text{complex conjugates} \end{cases}$

- $A(t)$ is complex, $\rightarrow$ $|A|e^{i\phi}$

$\rightarrow$ perturbation: $A(t)\, e^{i\vec{q}\cdot\vec{x}_\perp} = |A|\, e^{i\vec{q}\cdot\vec{x} + i\phi} = |A|\, e^{i\vec{q}\cdot(\vec{x} + \phi\frac{\vec{q}}{q})}$

phase: position of the pattern.

(Ex: Stationary instability: $\dfrac{dA}{dt} = \sigma A$

$$\rightarrow \frac{d|A|}{dt} e^{i\phi} + |A| e^{i\phi} \frac{d\phi}{dt} i = \sigma |A| e^{i\phi} \Rightarrow \begin{cases} \dfrac{d|A|}{dt} = \sigma |A| \\ \dfrac{d\phi}{dt} = 0 \end{cases}$$

)

Nonlinear stripe state of Swift-Hohenberg equation

- 1D Swift-Hohenberg equation: $\frac{\partial u}{\partial t} = ru - (\frac{\partial^2}{\partial x^2}+1)^2 u - u^3$

  growth rate: $\sigma_q = r^2 - (q-1)^2$

- $q = 1 = q_c$, $\sigma_q = r$ Max Re (σ_q) point.

  $\rightarrow$ grow mode: $e^{rt}\cos x$

  Assume steady saturated solution: $u = a_1 \cos x$

  $[\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos(3x)]$

  $\rightarrow 0 = (ra_1 - \frac{3}{4}a_1^3)\cos x - \frac{1}{4}a_1^3 \cos(3x)$

  $$\Rightarrow \begin{cases} ra_1 - \frac{3}{4}a_1^3 = 0 \\ \frac{1}{4}a_1^3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0,\ \pm\sqrt{\frac{4}{3}r} \\ a_1 = 0 \end{cases}$$

  $\Rightarrow$ no balance for nonlinear term

  $\rightarrow$ Assume: $a_1\cos x + a_3\cos(3x)$

  $$\begin{Bmatrix} \cos^5 x = \frac{1}{16}[10\cos x + 5\cos(3x) + \cos(5x)] \\ \cos^7 x = \frac{1}{64}[35\cos x + 21\cos(3x) + 7\cos(5x) + \cos(7x)] \end{Bmatrix}$$

  $$\begin{aligned} \rightarrow 0 = & \boxed{\begin{aligned} &(ra_1 - \tfrac{3}{4}a_1^3 + \tfrac{3}{4}a_1^2 a_3 + \tfrac{3}{2}a_1 a_3^2)\cos x \\ &+ (ra_3 - 64a_3 - \tfrac{1}{4}a_1^3 + \tfrac{3}{2}a_1^2 a_3)\cos(3x) \end{aligned}} \\ & + (\tfrac{3}{4}a_1^2 a_3 + \tfrac{3}{4}a_1 a_3^2)\cos(5x) \\ & + \tfrac{3}{4}a_1 a_3^2 \cos(7x) \\ & + \tfrac{1}{4}a_3^2 \cos(9x) \end{aligned}$$

  $$\Rightarrow \begin{cases} a_1 \sim O(r^{1/2}) \\ a_3 \sim O(r^{3/2}) \\ a_1^2 a_3 \sim O(r^{5/2}) \\ a_1 a_3^2 \sim O(r^{7/2}) \end{cases}$$

  $\rightarrow$ Need to add more for $\cos(5x)$. $\cos(7x)$. $\cos(9x)$

  $\Rightarrow$ solution: $u = \sum_{n=\text{odd}} a_n \cos(nx)$, $a_n \sim O(r^{n/2})$ for small r

  ex: $u = \pm\sqrt{\frac{4}{3}r}\cos x + O(r^{3/2})\cos(3x)$

  pattern grows in magnitude from $r^{1/2}$

- Numerical: Galerkin method, truncate to finite number of basis.

Stability balloons

- what analyze when $\vec{u}_b$ is spacially periodic in $\vec{x}_\perp$
  [Felix Bloch. 1928. Solve linear evolution equations with spacially periodic coefficients]

  For perturbation, solution : $e^{\sigma(\vec{Q},\vec{q})t}\, e^{i\vec{Q}\vec{x}_\perp}\, \vec{u}_{\vec{Q}}(\vec{x}_\perp, \vec{x}_\parallel)$ [Bloch state]

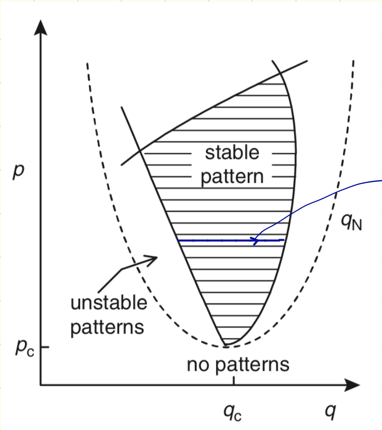
  $\begin{cases} \vec{Q} : \text{wave vector} \\ \vec{u}_{\vec{Q}} \text{ has same periodicity as base } \vec{u}_b(\vec{q}) \end{cases}$

  In 2D: $|||||$ y, x $(\vec{q})$ ✔ $\vec{Q}(Q_x, Q_y)$, $\vec{u}_{\vec{Q}}$ periodic in x, period $\frac{2\pi}{q}$

+ Want to know if $Re(\sigma(\vec{Q},\vec{q}))$ becomes positive.
  $\vec{u}_{\vec{Q}}$ is periodic in x with q $\rightarrow$ $e^{imqx}\vec{u}_{\vec{Q}}$ (any integar m)
+ Set range. $-\frac{q}{2} < Q_x < \frac{q}{2}$ $\rightarrow$ define $\vec{u}_{\vec{Q}}$: $e^{iqx}\vec{u}_{\vec{Q}}$

  $\begin{cases} \vec{Q} \text{ is zero : perturbation doesn't change periodicity of pattern} \\ Q_x = \frac{q}{2} \text{ (on the boundary): spatial period in } x \text{ is doubled.} \\ \text{instability may occur at longwave length } \vec{Q}=0 \end{cases}$

+ Conceptual: identify instability type.
  → each q, p s.t nonlinear staturated stripe state exist
  → find max $Re[\sigma(\vec{Q};q)]$ over all $\vec{Q}$
  → find $\max_{\vec{Q}} Re[\sigma(\vec{Q};q)] < 0$



Slowly growing domain
fixed number of stripes
(compressed / stretched)
eg: stripes of fish
grow bigger as age

[M. Cross and H. Greenside. 2009]

For stripe state (Eckhaus instability):
long wavelength longitudinal perturb. Q_x small, $Q_y = 0$)
zigzag instability:
long wavelength transverse distortion $Q_x = 0$, Q_y small)

Zigzag instability for Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} = ru - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 1\right)^2 u - u^3$$

+ stationary nonlinear stripe solution: $u_q(x) \approx a_q \cos(qx)$

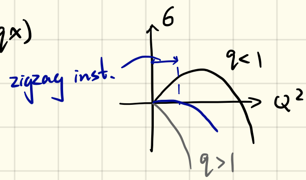
$\rightarrow a_q^2 = \frac{4}{3}\left[r - (q^2-1)^2\right]$

$\rightarrow$ stripe exists: $\frac{4}{3}\left[r - (q^2-1)^2\right] \geq 0$

$\rightarrow \sqrt{1-\sqrt{r}} \leq q \leq \sqrt{1+\sqrt{r}}$

transverse perturbation $\vec{Q} = Q\vec{y}$

$\rightarrow$ perturbation: $\begin{cases} \text{even: } \delta u = e^{\sigma t} e^{iQy} \sum_{n=1}^{\infty} C_n \cos(nqx) \\ \text{odd: } \delta u = e^{\sigma t} e^{iQy} \sum S_n \sin(nqx) \end{cases}$



$\rightarrow \sigma = 2(1-q^2)Q^2 - Q^4$ for odd

$= Q^2\left[2(1-q^2) - Q^2\right]$

+ If $q<1$, $\sigma > 0$ instability
stripes with $q<1$ are unstable to odd transverse perturb.

If $q \geq 1$, $\sigma < 0$ stability
stripes with $q \geq 1$ are stable to some perturb.

+ Zigzag instability: $(q<1)$ long wavelength small Q.

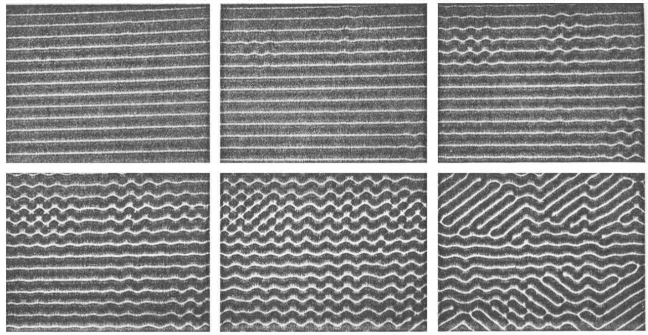
Limitation of stability balloon:

- Idealized infinity boundary.
- Tiny perturbation

Bass balloon for Rayleigh-Benard convection
[Fritz Bass 1965] [Galerkin method]

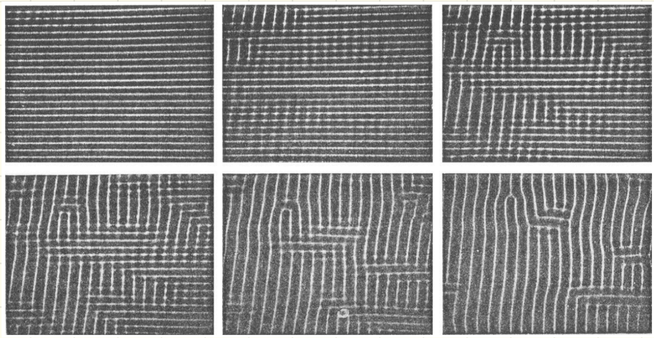
Prandle number $\sigma = 1 \times 10^2$

$Ra \approx 2R_c$
initial $q = 2.8$



The stripes are unstable to long wavelength transverve zigzag instability.

$Ra \approx 1.7\, R_c$
$q = 1.64$



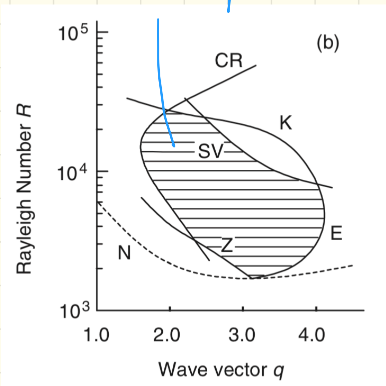
unstable to cross-roll instability

Stability balloon for stripes state [M. Cross and H. Greenside. 2009]

Prandtl number = 7.

Temperature of water = 40°C

stable pattern



CR: cross-roll (stationary, long wavelength, transverse)

K : knot (stationary, finite wavelength, transverse)

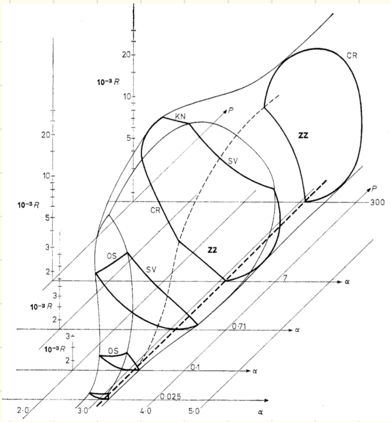
SV : skew-varicose (stationary, long wavelength, skew)

Z : Zigzag (stationary, long wavelength, transverse)

E : Eckhaus (stationary, long wavelength, longitudinal)

[F. H. Busse 1978]

Ra



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