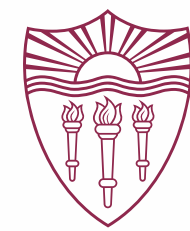


# PDE - constraint optimization

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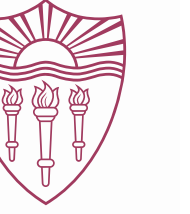
University of Southern California



Kanso's Bio-inspired Lab

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# Forward and Inverse Problem



## Forward problem

$$\mathcal{F}(c; \alpha) = 0$$

$\mathcal{F}(c; \alpha)$  represents a PDE operator acting on  $c$   
 $\alpha$  is a coefficient

## Inverse problem

$$\text{find } \alpha, \min_{\alpha} f(c(\alpha))$$

$$\text{s.t. } \mathcal{F}(c; \alpha) = 0$$

$f$ : objective function (*ie* .  $|c - c_o|^2$ )

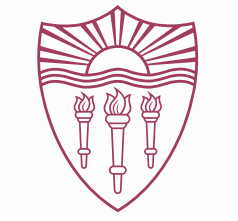
$\alpha$ : design variable

$c$ : state variable

<https://www.cs.cmu.edu/~ateh/presentations/slides-pde-constrained-optimization.html>

[https://kailaix.github.io/ADCME.jl/stable/tu\\_optimization/](https://kailaix.github.io/ADCME.jl/stable/tu_optimization/)

# Find adjoint variable



$$\text{find } \alpha, \min_{\alpha} f(c(\alpha))$$

$$\text{s.t. } \mathcal{F}(c; \alpha) = 0$$

$f$ : objective function, (ie.  $|c - c_o|^2$ )

$\alpha$ : design variable

$c$ : state variable

we want to use gradient based optimization,

we need,  $\partial_{\alpha} f(c(\alpha))$

$$\partial_{\alpha} f = \partial_c f \boxed{\partial_{\alpha} c} \quad \text{Obtain by using PDE}$$

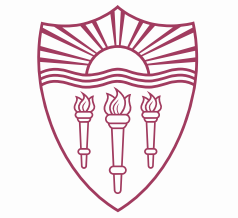
from  $\mathcal{F}(c(\alpha); \alpha) = 0$ ,

$$\text{we have } d_{\alpha} \mathcal{F} = \partial_c \mathcal{F} \partial_{\alpha} c + \partial_{\alpha} \mathcal{F} = 0$$

$$\rightarrow \partial_{\alpha} c = - (\partial_c \mathcal{F})^{-1} \partial_{\alpha} \mathcal{F}$$

$$\partial_{\alpha} f = \lambda^T \partial_c f, \quad \text{where } \lambda^T = - (\partial_c \mathcal{F})^{-1} \partial_{\alpha} \mathcal{F}$$

# Link to Lagrange function



find  $\alpha$ ,  $\min_{\alpha} f(c(\alpha))$

s . t .  $\mathcal{F}(c; \alpha) = 0$

$f$  : objective function, (ie .  $|c - c_o|^2$ )

$\alpha$  : design variable

$c$  : state variable

we want to use gradient based optimization,  
we need,  $\partial_{\alpha} f(c(\alpha))$

Lagrange function,  $L(c, \alpha, \lambda) = f(c(\alpha)) + \lambda^T \mathcal{F}(c; \alpha)$

KKT conditions :

$$(1) \quad \partial_{\lambda} L(c, \alpha, \lambda) = \mathcal{F}(c; \alpha) = 0$$

$$(2) \quad \partial_c L(c, \alpha, \lambda) = \partial_c f(c) + \lambda^T \partial_c \mathcal{F}(c; \alpha) = 0$$

$$(3) \quad \partial_{\alpha} L(c, \alpha, \lambda) = \partial_{\alpha} f(c) + \lambda^T \partial_{\alpha} \mathcal{F}(c; \alpha) = 0$$

$$\partial_{\alpha} f = \partial_c f \partial_{\alpha} c$$

Obtain by using PDE

from  $\mathcal{F}(c(\alpha); \alpha) = 0$ ,

$$\text{we have } d_{\alpha} \mathcal{F} = \partial_c \mathcal{F} \partial_{\alpha} c + \partial_{\alpha} \mathcal{F} = 0$$

$$\rightarrow \partial_{\alpha} c = - (\partial_c \mathcal{F})^{-1} \partial_{\alpha} \mathcal{F}$$

$\partial_{\alpha} f = \lambda^T \partial_c f$ , where  $\lambda^T = - (\partial_c \mathcal{F})^{-1} \partial_{\alpha} \mathcal{F}$ , is adjoint variable

for a given  $\alpha$ , solve for  $c, \lambda$ , from(1), (2),

$$\rightarrow \partial_{\alpha} L(c, \alpha, \lambda) = \partial_{\alpha} f - \partial_c f (\partial_c \mathcal{F})^{-1} \partial_{\alpha} \mathcal{F}$$

When alpha is optimal, this expression is equal to zero, i.e., all the KKT conditions are satisfied.